FINITE GROUPS NOT HAVING THE BORSUK-ULAM PROPERTY

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ABSTRACT. Determination of finite groups with the Borsuk-Ulam property is one of classical problems in the study of Borsuk-Ulam type theorems. Many relevant results have been shown so far, and recently, we succeeded in providing a complete answer to this problem. In this article, we shall provide an outline of the proof and some remarks on the Borsuk-Ulam constant.

1. BORSUK-ULAM PROPERTIES AND THE MAIN STATEMENT

Let G be a finite group and all maps between spaces are assumed to be continuous. For a unitary or orthogonal G-representation V, we denote the unit sphere by $S(V)$, called the representation sphere. Moreover we assume that representations are fixed-point-free, i.e., $V^G = 0$ unless otherwise stated.

K. Borsuk [3] proved the so-called Borsuk-Ulam theorem, which is described in various way. For example, the following is known.

Proposition 1.1. *The following statements hold.*

(1) *For any map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.*
(2) *There is no odd map $f : S^n \to S^{n-1}$.*
(3) *If $h : S^{n-1} \to S^{n-1}$ is an odd map, then $\deg h \neq 0$, i.e., f is not null-homotopic.*

Remark. Furthermore, $\deg h$ is odd for any odd map $h : S^{n-1} \to S^{n-1}$.

Many generalizations of the Borsuk-Ulam theorem have been studied; for example, the following result was shown in 1980's.

Proposition 1.2. *Let $G = C_p{}^k$ or $(S^1)^k$ (p: prime, $k \geq 0$).*

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(1) *If there exists a G-map* $f : S(V) \to S(W)$, *then* $\dim V \leq \dim W$.
(2) *If there exists a G-map* $h : S(V) \to S(W)$ *with* $\dim V = \dim W$, *then* $\deg h \neq 0$.

On the other hand, T. Bartsch [1] and W. Marzantowicz [7] have shown that Borsuk-Ulam type results as in Proposition 1.2 do not hold for many finite groups. This leads us to the following definition and problem.

Definition. Let G be a finite group.

(1) We say that G has the *Borsuk-Ulam property of type* (I) (BUP (I) for short) if $\dim V \leq \dim W$ holds whenever there exists a G-map $f : S(V) \to S(W)$.
(2) We say that G has the *Borsuk-Ulam property of type* (II) (BUP (II) for short) if $\deg h \neq 0$ holds whenever there exists a G-map $h : S(V) \to S(W)$ with $\dim V = \dim W$.

Problem. *Determine finite groups with BUP (I) [resp. (II)].*

A complete answer to this problem is given as follows.

Theorem 1.3 ([10])**.** *The following statements are equivalent.*

(1) *G has BUP (I).*
(2) *G has BUP (II).*
(3) *G is an elementary abelian p-group ${C_p}^k$, $k \geq 0$.*

In the following sections, we give an outline of the proof of this theorem. The details are described in [10].

2. Reduction to the case of M_p

The Borsuk-Ulam theorem does not hold for almost finite groups, more precisely, the following is known.

Proposition 2.1 ([1], [7])**.** (1) *Non p-groups have neither BUP (I) nor BUP (II).*
(2) *Finite groups with an element of order p^2 have neither BUP (I) nor BUP (II).*

Therefore the remaining groups are finite p-groups of exponent p. If G is abelian, then such G is isomorphic to an elementary abelian p-group. If $|G| \leq p^2$ or $p = 2$, then G is abelian and hence an elementary abelian p-group. Therefore we may consider only non-abelian finite p-groups of exponent p, where p is an odd prime. For any odd prime p, if $|G| = p^3$, then there exists only one non-abelian p-group M_p of exponent p:

$$M_p = \langle a, b, c \,|\, a^p = b^p = c^p = 1, ac = ca, bc = cb, a^{-1}ba = bc \rangle,$$

see [6]. We would like to show that non-abelian finite p-groups of exponent p have neither BUP (I) nor BUP (II). The following proposition shows that the problem is reduced to the case of M_p.

Proposition 2.2 ([10])**.** *If M_p does not have BUP (I) [resp. (II)], then every non-abelian p-group of exponent p does not have BUP (I) [resp. (II)].*

This will be proved by using the following basic property.

Proposition 2.3 (Basic property [10])**.** *If G has BUP (I) [resp. (II)], then*

(1) *any subgroup H has BUP (I) [resp. (II)], and*
(2) *any quotient group $Q = G/K$ has BUP (I) [resp. (II)].*

Outline of the proof of Proposition 2.2. It is shown by induction on $k \geq 3$ ($|G| = p^k$). If $k = 3$, then $G \cong M_p$, hence it is true by assumption.

Since G is non-abelian, there exists a unitary irreducible representation U such that $\dim_{\mathbb{C}} U \geq 2$. If $K := \operatorname{Ker} U \neq 1$, then $U = U^K$ is a unitary irreducible G/K-representation of $\dim_{\mathbb{C}} U^K \geq 2$. Therefore G/K is non-abelian and of exponent p. By inductive assumption, G/K does not have BUP (I) [resp. (II)], hence G does not have BUP (I) [resp. (II)] by the basic property.

If $K = 1$, then U is faithful, and hence the center $Z(G)$ is cyclic by representation theory. Therefore we have $Z(G) \cong C_p$. If $G/Z(G)$ is non-abelian, then G does not have BUP (I) [resp. (II)] by the basic property. If $G/Z(G)$ is abelian, then we have $G/Z(G) \cong {C_p}^{k-1}$. Thus G is an extra-special p-group. By [6], an extra-special p-group G of exponent p has a subgroup isomorphic to M_p. This shows that G does not have BUP (I) [resp. (II)] by the basic property. □

3. Construction of counterexamples

In this section, we would like to construct counterexamples to BUP (I) and (II) for M_p.

To do that, we consider a compact Lie group $\widetilde{M_p}$ including M_p as a subgroup:

$$\widetilde{M_p} = \langle a, b, \zeta \mid a^p = b^p = 1, \zeta a = a\zeta, \zeta b = b\zeta \ (\forall \zeta \in S^1), a^{-1}ba = b\xi_p \rangle,$$

where $\xi_p = \exp(2\pi\sqrt{-1}/p)$. We first construct a $\widetilde{M_p}$-map $h : S(V) \to S(W)$ of degree 0 for some fixed-point-free $\widetilde{M_p}$-representations V and W. By restricting to M_p, we obtain a counterexample to BUP (II). We next construct a $\widetilde{M_p}$-map $f : S(V) \to S(W)$ for some V and W with $\dim V > \dim W$ using a degree 0 $\widetilde{M_p}$-map h. By restricting to M_p, we will obtain a counterexample to BUP (I).

We consider the following irreducible (unitary) $\widetilde{M}_p$-representations $V_{k,l}$, $(k,l) \in \mathbb{F}_p^2 \setminus \{(0,0)\}$ and U_m, $m \in \mathbb{F}_p^*$:

(1) $V_{k,l}$ is 1-dimensional and its character is given by $V_{k,l}(a) = \xi_p^k$, $V_{k,l}(b) = \xi_p^l$ and $V_{k,l}(\xi) = 1$.

(2) U_m is p-dimensional and its character is given by $U_m(g) = \begin{cases} \xi_p^{mu} & \text{if } g = \xi_p^u \\ 0 & \text{otherwise.} \end{cases}$

Restricting $V_{k,l}$ and U_m to M_p, we obtain fixed-point-free M_p-representations.

Set

$$V = 2U_1 \quad \text{and} \quad W = V_{0,p-1} \oplus 2V_{1,p-1} \oplus \cdots \oplus 2V_{p-1,p-1} \oplus V_{1,0}.$$

Note that $\dim V = \dim W$. By equivariant obstruction theory [4], [5], there exists an $\widetilde{M}_p$-map $h : S(V) \to S(W)$. Then we have

Proposition 3.1. $\deg h = 0$. *Hence M_p does not have BUP (II).*

Proof (Sketch). We use the Euler class

$$e(V) := e(E\widetilde{M}_p \times_{\widetilde{M}_p} V) \in H^{2n}(B\widetilde{M}_p; \mathbb{Q})$$

of an oriented representation V with $\dim V = 2n$. A unitary representation V naturally become an oriented representation and hence the Euler class of V is defined. In general, by [8], if there exists a G-map $h : S(V) \to S(W)$ for fixed-point-free representations with the same dimension, then $e(W) = (\deg h)e(V)$ holds.

By an argument of the Serre spectral sequence, we see that

$$\text{Res} : H^*(B\widetilde{M}_p; \mathbb{Q}) \to H^*(BS^1 \, Q) \cong \mathbb{Q}[t]$$

is a ring isomorphism. Using this isomorphism, we see $e(V) \neq 0$ and $e(W) = 0$ for the representations defined as above. Hence we obtain $\deg h = 0$. $\square$

Let V and W be as before. Set

$$\widetilde{V} = 2V \oplus V_{1,p-1}, \quad W_1 = V \oplus W \quad \text{and} \quad \widetilde{W} = 2W.$$

Note $\dim S(\widetilde{V}) = 8p+1$ and $\dim S(W_1) = \dim S(\widetilde{W}) = 8p-1$. By an obstruction theoretic argument, we see that there exits an $\widetilde{M}_p$-map $f_{(8p)} : S(\widetilde{V})_{(8p)} \to S(W_1)$, where $S(\widetilde{V})_{(8p)}$ is the $8p$-skeleton of $S(\widetilde{V})$. We would like to extend $f_{(8p)}$ on $S(\widetilde{V})$, however, there is an obstruction to do it in general. In order to avoid this difficulty, we consider a degree 0 $\widetilde{M}_p$-map $\widetilde{h} := h * id : S(W_1) \to S(\widetilde{W})$. Since the composite map $\widetilde{h} \circ f_{(8p)}$ is null-homotopic, we can extend $\widetilde{h} \circ f_{(8p)}$ on $S(\widetilde{V})_{(8p+1)} = S(\widetilde{V})$. Thus we obtain an $\widetilde{M}_p$-map $f : S(\widetilde{V}) \to S(\widetilde{W})$, which provides a counterexample to BUP (II).

4. From the point of view of the Borsuk-Ulam constant

Bartsch [2] and Meyer [9] introduced the following constant a_G which we call the *(equivariant) Borsuk-Ulam constant* of G.

Definition. $a_G = \sup\{a \in \mathbb{R} \mid \text{ a constant } a \text{ satisfies condition (WBU)}\}$.

(WBU): $a \dim V \leq \dim W$ holds whenever there exists a G-map $f : S(V) \to S(W)$.

Proposition 4.1. *The hollowing hold.*

(1) $a_G = \inf\{\dim W / \dim V \mid$ *there exists a* G*-map* $f : S(V) \to S(W)\}$.

(2) $0 \leq a_G \leq 1$.

(3) $a_G = 1$ *if and only if* G *has BUP (II).*

Theorem 1.3 shows that $a_G = 1$ if and only if G is an elementary abelian group. By a result of [1], if G is not a p-group, then $a_G = 0$. When G is a p-group, strict values of a_G are not known other than cyclic p-groups. Meyer [9] (and also Bartsch [2]) showed $a_G = 1/p^{k-1}$ for $G = C_{p^k}$.

Bartsch conjectures in [2] that $a_G = 1/p^{k-1}$ for a p-group G of exponent p^k. However our result shows that this conjecture is false. Indeed, the exponent of $G = M_p$ is p and the conjecture asserts that $a_G = 1$, however, we see $a_G \leq \frac{4p}{4p+1}$ by an existence of a G-map $\operatorname{Res} f : S(\operatorname{Res} \widetilde{V}) \to S(\operatorname{Res} \widetilde{W})$ in section 3.

We finally pose some unsolved problems.

Problem. (1) *Give a lower estimate of* a_{M_p}.

(2) *Is it true that* $a_G > 0$ *for a* p*-group* G*?*

Remark. If $a_G > 0$, the weak Borsuk-Ulam theorem in the sense of Bartsch [1]. In finite group case, Bartsch shows that the weak Borsuk-Ulam theorem holds if and only if G is a p-group. Therefore, if problem (2) is true, then the weak Borsuk-Ulam theorem holds if and only if $a_G > 0$.

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